

The quantum mechanical tipping pencil—a caution for physics teachers

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Abstract

Since at least as long ago as 1960, quantum mechanics has been applied with varying degrees of rigour to the problem of a pencil balancing on its point. Although the analysis of the problem may have some merit in terms of solving for the evolution of and the final state of a similarly shaped (rod-like) but microscopic object, it is misleading when students are given the idea that quantum mechanics is useful in describing the behaviour of macroscopic objects.

Recently, while I was reworking some problems in Dicke and Witke's *Quantum Mechanics* [1], I became intrigued by a problem that I remember well from my introductory course in quantum mechanics taken while I was a student at the University of Alberta. At the time, I neither solved the problem nor understood why it was included in a text on quantum mechanics. Simply stated, the problem asks how long can a pencil be balanced on its point?

A naïve solution

When I attempted the question, I treated it as a Fermi problem. My quick estimate was based on the reasoning that in order to be balanced, the centre of mass of the pencil must be confined to a region directly above the small, but not infinitesimally small, point of the pencil (figure 1). In a very naïve way, I applied the uncertainty principle. I did not do this because I thought it to be an excellent way to solve the problem, but because it is straightforward and ought to give an order of magnitude estimate for the tipping time.

Imagining the pencil ($m = 0.01$ kg) to be well sharpened so that the point forms a base with a width on the order of an atomic dimension, set Δx to be approximately 10^{-10} m. The uncertainty principle can be applied in the form

$$\Delta p \Delta x \geq \frac{h}{4\pi}.$$

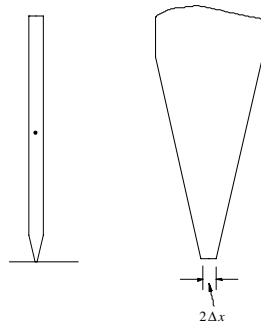


Figure 1. A pencil is balanced on its point when the centre of mass is directly over the sharpened point. The point is small, but has a finite width $2\Delta x$.

This does not take into account the acceleration of gravity, g , but that is acceptable since the motion of the pencil's centre of mass is very nearly horizontal while the centre of mass is in the small region $2\Delta x$. The value of g determines the topple time after the centre of mass leaves the 'safe' zone.

Substituting values for m and Δx leads to

$$\Delta v \geq 5 \times 10^{-23} \text{ m s}^{-1}.$$

The maximum time that the centre of mass can be above the base is on the order of $2\Delta x/\Delta v \approx 4 \times 10^{12}$ s. In a large number of balancing experiments, one might expect the average balancing time to be half this amount or about 0.6 million years.

This estimate is not anything like my experience with real pencils, so I concluded that the intent of the Dicke–Witke problem was to show that the tipping of a pencil is not a quantum mechanical effect.

I was about to close the books on this problem, but decided to perform a quick internet search to see if there were any comments on the balancing pencil problem. It was surprisingly easy to find posted solutions (deliberately unreferenced). It was even more surprising that the solutions indicated a maximum quantum mechanical balancing time on the order of 3 s.

An incorrect solution

The posted analyses were essentially the same. I reproduce them here using my own notation, but maintaining the essential features of the original sources. Referring to figure 2, treat the pencil as an inverted pendulum of mass m . The distance from the pivot to the centre of mass is $L/2$ and the axis of the pencil makes an angle, θ , with the vertical. For small angles, the equation of motion for the pencil is

$$\frac{d^2\theta}{dt^2} + \frac{2g}{L}\theta \approx 0. \quad (1)$$

This has the solution

$$\theta = A e^{\sqrt{2g/L}t} + B e^{-\sqrt{2g/L}t}, \quad (2)$$

in which A and B are small (in this case) constants determined by the initial conditions.

Writing the initial conditions as

$$\theta(0) = \theta_0 \quad (3a)$$

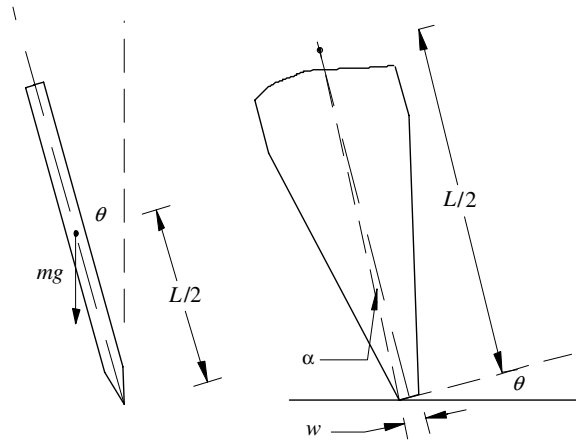


Figure 2. This figure shows the essential features of the pencil standing on its point. Because of the finite width of the point, the centre of mass of the pencil rises slightly when the pencil tips away from the vertical. The vertically oriented pencil resides at the local minimum of the potential energy.

and

$$\frac{d\theta}{dt}(0) = \omega_0 \tag{3b}$$

gives

$$\theta = \frac{1}{2}(\theta_0 + \omega_0\sqrt{L/2g}) e^{\sqrt{2g/L}t} + \frac{1}{2}(\theta_0 - \omega_0\sqrt{L/2g}) e^{-\sqrt{2g/L}t}. \tag{4}$$

The internet solutions incorrectly suggest that the growth of the positive exponential necessarily overwhelms the decaying exponential so that for any reasonably large θ ($\theta \gg \theta_0$), only the first term needs to be considered:

$$\theta \approx \frac{1}{2}(\theta_0 + \omega_0\sqrt{L/2g}) e^{\sqrt{2g/L}t}. \tag{5}$$

The uncertainty principle [2]¹ in the form

$$\Delta p_\theta \Delta \theta \geq \frac{h}{4\pi} \tag{6}$$

is used, to give $\omega_0 \geq 2h/\pi m L^2 \theta_0$. This results in

$$\theta \approx \frac{1}{2} \left(\theta_0 + \sqrt{2h^2/\pi^2 m^2 L^3 g} \frac{1}{\theta_0} \right) e^{\sqrt{2g/L}t}. \tag{7}$$

By taking the derivative with respect to θ_0 , a minimum value for the coefficient can be determined. The minimum value occurs at

$$\theta_0 = \sqrt[4]{\frac{2h^2}{\pi^2 m^2 L^3 g}}. \tag{8}$$

Equation (8) can be substituted into equation (7) to give

$$\theta = \sqrt[4]{\frac{2h^2}{\pi^2 m^2 L^3 g}} \cdot e^{\sqrt{2g/L}t}, \tag{9}$$

¹ This article contains an interesting discussion of the inappropriateness of using the uncertainty in the tipping pencil problem.

which can be solved for t . The result is

$$t = \sqrt{\frac{L}{2g}} \left[\frac{1}{4} \ln \left(\frac{\pi^2 m^2 L^3 g}{2h^2} \right) + \ln \theta \right]. \quad (10)$$

The angle, θ , is arbitrarily chosen to be the angle at which the pencil ‘... appears to be unbalanced.’ For a typical pencil ($m = 0.01$ kg, $L = 0.2$ m), setting $\theta = 0.1$ results in a maximum balancing time of ~ 3 s.

One author makes the statement that the ‘... smallness of this answer is quite surprising. It is remarkable that a quantum effect on a macroscopic object can produce an everyday value for a time scale’. It should be surprising; it is incorrect. The analysis fails on at least two counts.

First, the object being analysed is ostensibly a macroscopic object, but in fact it is not truly macroscopic. The pencil has been sharpened to a point of infinitesimal width. It is inappropriate to discuss how long such a pencil can remain balanced. It is never balanced. In the physicist’s world of imaginary things, the pencil is ‘balanced’ in unstable equilibrium when the centre of mass is directly above the pencil point. However, in the real world, unstable equilibrium is as close to unbalanced as dammit is to swearing. If the pencil is to be truly balanced on its point, the point must have a finite width. The nearly vertically oriented pencil must reside in a potential energy well; there must be a restoring torque.

The second problem arises from the elimination of the decaying exponential in equation (4). Because ω_0 does not necessarily have the same sense as θ_0 —the uncertainty in the angular momentum is in part due to the uncertainty in its direction—it is possible for the coefficient of the positive exponential to be zero and the decaying exponential becomes the dominant (only) term. The behaviour of the pencil can then be found by considering a negative (opposite sense to θ_0) value for the initial angular velocity, $\omega_0 = -2h/\pi mL^2\theta_0$. In this situation, the motion of the pencil is described by

$$\theta = \sqrt[4]{\frac{2h^2}{\pi^2 m^2 L^3 g}} e^{-\sqrt{2g/L}t}. \quad (11)$$

This describes a dynamically balanced pencil that moves towards a vertical orientation with ever decreasing speed. Now the maximum time for which the pencil lingers near its upright position is infinite.

A second naïve solution

To treat the pencil as a macroscopic object, consider the pencil point to have a finite width, w , and note that tipping the pencil away from the vertical slightly raises the centre of mass (figure 2). The potential energy depends on the cosine of the angle $\theta - \alpha$, where θ is the angle that the pencil makes with the vertical and $\alpha = w/L$. Since the angles are both small while the pencil is balanced, the potential energy can be approximated by the parabolic function

$$V = -\frac{mgL}{4} \left(\theta - \frac{w}{L} \right)^2 + \frac{mgw^2}{4L}. \quad (12)$$

A plot of potential energy versus θ (figure 3) shows that there is a potential well that classically would contain the pencil so long as its energy remains less than the barrier height. Quantum mechanically, there is a small probability of a tunnelling event; the pencil could fall over.

The naïve solution that I propose is to approximate the potential energy function as a square well and barrier (figure 4). The square well representation is chosen because it is

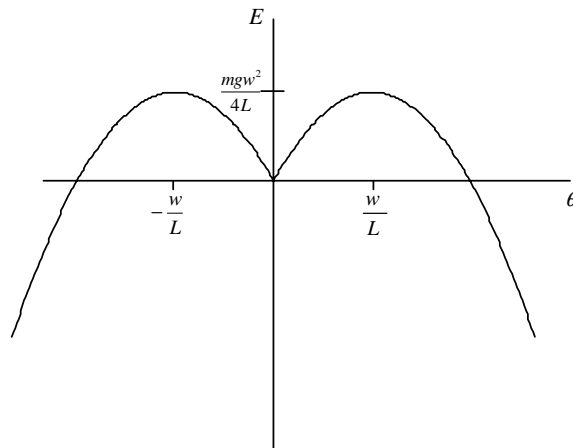


Figure 3. The potential energy as a function of the angle that the pencil makes with the vertical comprises two parabolic sections.

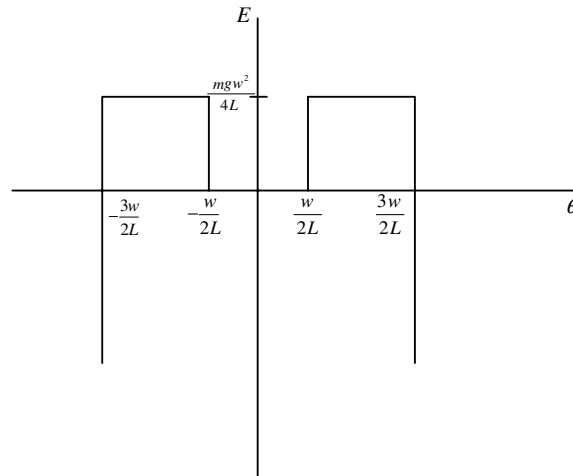


Figure 4. The potential energy can be modelled as a square well bounded by two square barriers.

solvable by elementary means. The solutions for square wells and barriers are not repeated here since they are well represented in introductory texts on quantum mechanics [1, 3]. The essential features of a Schrodinger wave equation solution is that the wavefunctions (ψ 's) are oscillatory in the regions $-w/2L < \theta < w/2L$ and $|\theta| > 3w/2L$, but exponentially decaying functions in the regions $w/2L < |\theta| < 3w/2L$.

The wavefunctions and their first derivatives with respect to θ must be continuous at the boundaries $\theta = \pm w/2L$ and $\theta = \pm 3w/2L$. Applying the boundary conditions leads to the solution

$$\psi_1 = 2A \cos k_1 \theta, \tag{13a}$$

$$\psi_2 = 2 e^{k_2 \frac{w}{2L}} \cos \left(k_1 \frac{w}{2L} \right) A e^{-k_2 \theta} \tag{13b}$$

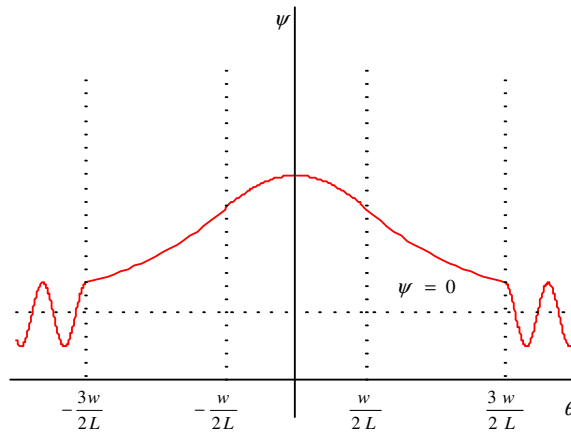


Figure 5. The wavefunction representation is oscillatory inside and outside the potential well, but decays exponentially within the barrier—a classically forbidden region. In that region, the total energy of the pencil is less than its potential energy.

and

$$\psi_3 = F \cos(k_3\theta - \delta) = 2 e^{-k_2 \frac{w}{L}} \frac{\cos(k_1 \frac{w}{2L})}{\cos(\frac{k_2}{k_3})} A e^{-k_2\theta}. \tag{13c}$$

k 's are given by

$$k_1 = \sqrt{\frac{2mL^2}{3h^2} (E - 0)}, \tag{14a}$$

$$k_2 = \sqrt{\frac{2mL^2}{3h^2} \left(\frac{mgw^2}{4L} - E \right)} \tag{14b}$$

and

$$k_3 = \sqrt{\frac{2mL^2}{3h^2} \left(E + \frac{mgw^2}{L} \right)}. \tag{14c}$$

The value of the potential energy term in k_3 was arbitrarily chosen to represent a substantial loss of potential energy, while keeping the pencil in an orientation not too different from its upright position.

To meet the requirements of the boundary conditions, the energy of the pencil and the associated k values take on discrete values. Figure 5 shows the general features of the lowest energy solution—the shapes are right, but the amplitudes do not accurately reflect the behaviour of a pencil. The actual amplitudes of the two oscillatory solutions cannot be drawn on the same graph; they are different by many orders of magnitude.

Classically, the pencil is balanced and will not fall over unless there is an input of energy that is sufficient to overcome the potential barrier. Quantum mechanically, there is a probability of a tunnelling event. The pencil could fall over. The probability of tunnelling is represented by a transmission coefficient

$$T = \frac{k_3 |F|^2}{k_1 |A|^2}. \tag{15}$$

Using values for a ‘well-sharpened’ pencil of ordinary dimensions $L = 0.2$ m, $m = 0.01$ kg and $w = 2 \times 10^{-10}$ m yields a transmission coefficient of $T = 10^{-10^{14}}$ which is a very small number. The probability of a tunnelling event is very nearly zero; the barrier is not much different from the classical brick wall.

With such a low probability of tunnelling, the frequency of the barrier encounters would have to be very high to produce a tipping time on an ordinary time scale. The time between encounters with the barrier can be found from

$$T = \frac{w/L}{\omega}, \quad (16)$$

where

$$\omega = \left[\frac{6E}{mL^2} \right]^{1/2}. \quad (17)$$

The result is $T \approx 2 \times 10^{12}$ s.

Conclusion

Considering the low probability of tunnelling and the low frequency of encounters with the barrier, one is forced to conclude that quantum mechanical effects are not responsible for the observed behaviour of pencils. We are quite safe from the intrusion of quantum mechanics into our everyday world of Newtonian objects. The 3 s quantum mechanical tipping of a pencil is an urban myth of physics. Physics teachers should approach this problem with caution to ensure that their students are not getting the wrong idea about quantum mechanics and the realm to which it applies.

Postscript

Thanks to the anonymous reviewer who has gently shown me that since neither classical nor quantum mechanics predicts that the pencil necessarily falls over, ‘... one is left believing that this system should remain in its nearly unstable equilibrium situation for a very long time.’ This is nothing like the observed behavior of pencils; a real pencil falls over in short order.

So, why does a pencil fall over? It falls over for the same reason that any similarly-shaped, macroscopic object would fall over. The well that constrains the center of mass of a pencil-shaped object is just not much of a well. We could imagine a telephone pole sharpened to a point a few centimetres across. It would not be a difficult task to set the telephone pole on its point in such a way that the centre of mass is over the few square centimetre area of the tip. We wouldn’t stand too close, however, because clearly the small amount of energy required to put the pole into its unstable orientation is readily available.

As we reduce the size of the object from telephone poles down toward pencils, two things happen. The width of the well is reduced and so is its depth. The result is that it becomes more difficult to place the centre of mass directly over the point and the energy required to nudge it out of the out of the well decreases. These two effects conspire to make the balancing of a pencil on its point essentially impossible. The idealized pencil that I described above has a point of atomic dimension and requires energy of the order 10^{-21} J to put it into an unstable orientation. The potential well is a very small target position and as for the amount of energy that will upset the balance—we live in a jiggly world.

References

- [1] Dicke R H and Wittke J P 1960 *Introduction to Quantum Mechanics* 1st edn (Reading, MA: Addison-Wesley) chapter 2, p 35
- [2] Shegelski M R A, Lundberg M and Goodvin G L 2005 *Am. J. Phys.* [73 686–9](#)
- [3] Anderson E E 1971 *Modern Physics and Quantum Mechanics* 1st edn (Philadelphia: Saunders) chapter 5, pp 167–72